Value-Positivity for Matrix Games

Stability in Matrix Games

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Classical problems. Matrix games and Linear Programming (LP).
Classical question. Stability:

How do our objects of interest change upon perturbations?

Observables. Solutions and value of the problems.

How do solutions and value change upon perturbations?

Matrix Games

$$\begin{array}{cc} & j \\ i & \left(\quad m_{i,j} \quad \right) \end{array} .$$

$$\operatorname{val} M := \max_{p \in \Delta[m]} \min_{q \in \Delta[n]} p^t M q \, .$$

Polynomial Matrix Games

$$M(\varepsilon) = M_0 + M_1\varepsilon + \ldots + M_K\varepsilon^K \,.$$

Definition (Value-positivity problem)

Determine if $\exists \varepsilon_0 > 0$ such that $\forall \varepsilon \in [0, \varepsilon_0]$

$$\mathrm{val} M(\varepsilon) \geq \mathrm{val} M(0) \,.$$

Polynomial Matrix Games II

Definition (Uniform value-positivity problem)

Determine if $\exists p_0 \in \Delta[m] \quad \exists \varepsilon_0 > 0$ such that $\forall \varepsilon \in [0, \varepsilon_0]$

$$\forall j \in [m] \quad (p_0^t M(\varepsilon))_j \geq \mathrm{val} M(0)\,.$$

Definition (Functional form problem)

Return the maps

$$\varepsilon \mapsto \mathrm{val} M(\varepsilon)$$
$$\varepsilon \mapsto p^*(\varepsilon)\,,$$

for $\varepsilon \in [0, \varepsilon_0]$.

Note. They are rational functions.

LPs with errors

An error-free LP is the following optimization problem.

$$(P_0)\begin{cases} \min_x & c_0^t x \\ s.t. & A_0 x \leq b_0 \\ & x \geq 0\,, \end{cases}$$

An LP with errors considers polynomials $A(\varepsilon), b(\varepsilon), c(\varepsilon)$.

Definition (Weak robustness)

Determine if there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in [0, \varepsilon_0]$, (P_ε) is feasible and bounded.

LPs with errors II

Definition (Strong robustness)

Determine if there exists $\varepsilon_0 > 0$ and a vector x_0^* such that, for all $\varepsilon \in [0, \varepsilon_0]$, x_0^* is also a solution of (P_ε).

Definition (Functional form)

Let $(P_\varepsilon)_{\varepsilon \geq 0}$ be weakly robust. The *functional form* is given by

$$\varepsilon \mapsto \mathrm{val}(P_\varepsilon)$$
$$\varepsilon \mapsto x^*(\varepsilon)\,.$$

Note. They are rational functions.

Polynomial matrix game

Consider $\varepsilon > 0$.

$$M(\varepsilon) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -3 \\ 0 & 2 \end{pmatrix} \varepsilon \,.$$

The optimal strategy is given by, for $\varepsilon < 1/2$,

$$p_\varepsilon = \left(\frac{1+\varepsilon}{2+3\varepsilon}, \frac{1+2\varepsilon}{2+3\varepsilon} \right)^t \,.$$

Therefore,

$$\mathrm{val} M(\varepsilon) = \frac{\varepsilon^2}{2+3\varepsilon} \,.$$

Polynomial matrix game, negative direction

Consider $\varepsilon > 0$.

$$M(\varepsilon) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 3 \\ 0 & -2 \end{pmatrix} \varepsilon \,.$$

The optimal strategy is given by, for $\varepsilon < 1/2$,

$$p_\varepsilon = \left(\frac{1+\varepsilon}{2+3\varepsilon}, \frac{1+2\varepsilon}{2+3\varepsilon} \right)^t \,.$$

Therefore,

$$\mathrm{val} M(\varepsilon) = \frac{\varepsilon^2}{2+3\varepsilon} \,.$$

LP with errors: not weakly robust

$$(P_\varepsilon)\begin{cases} \min_x & x \\ s.t. & x \leq -\varepsilon \\ & -x \leq -\varepsilon\,. \end{cases}$$

It is not weakly robust, therefore not strongly robust and there is no functional form.

LP with errors: strongly robust

$$
(P_\varepsilon) \begin{cases} \max_{x,y} & x + y \\ s.t. & x \leq 0 \\ & y + \varepsilon x \leq 0\,. \end{cases}
$$

It is weakly robust and strongly robust. Therefore, the functional form is:

$$
\begin{aligned}
\mathrm{val}(P_\varepsilon) &\equiv 0 \\
(x, y)^*(\varepsilon) &\equiv (0, 0)\,.
\end{aligned}
$$

Derivative of the value function [Mills 1956]

Define

$$D\mathrm{val}M(0^+) := \lim_{\varepsilon \to 0^+} \frac{\mathrm{val}M(\varepsilon) - \mathrm{val}M(0)}{\varepsilon} .$$

Results.

1. Characterization of $D\mathrm{val}M(0^+)$.
2. (Poly-time) algorithm for computing it.

Similar results are given for LPs with errors, adding feseability and boundedness conditions.

Observations.

1. Mills (naturally) only cares about linear perturbations.
2. This result does not solve in any way the value-positivity problems.

Reduction

Lemma (Reduction from LP with error to polynomial matrix games)

There is a polynomial-time reduction from robustness problems to the respective value-positivity problem, and it preserves the degree of the error terms.

Algorithms

Lemma (Poly-time algorithms)

There are certifying polynomial-time algorithms for value-positivity, uniform value-positivity and functional form problems of polynomial matrix games.

This extends the work of [Mills 1956].

LP reduction

Follows the work of [Adler 2013], which finishes the work of [Dantzig 1951]. Works for algebraic parameters.

Poly-time for value-positivity

Based on ideas of [Oliu-Barton 2020], which reuses results of [Shapley and Snow 1952].

Lemma

Let M be a matrix game. There exists a square submatrix $\dot{M}$ such that $\mathbb{1}^t co(\dot{M})\mathbb{1} \neq 0$ and

$$\mathrm{val} M = \frac{\det \dot{M}}{\mathbb{1}^t co(\dot{M})\mathbb{1}},$$

where $co(\dot{M})$ is the co-matrix of $\dot{M}$ and $\mathbb{1}$ is the vector of ones.

Arbitrary high degree for linear matrix games. The value function of a linear matrix game is a rational function

$$\mathrm{val} M(\varepsilon) = \frac{R_1(\varepsilon)}{R_2(\varepsilon)},$$

where R_1, R_2 are polynomials of degree at most $\min(\#rows, \#columns)$.
Is there a example where this inequality is sharp?

Your proposal?